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Motions of a particle on a sphere and integrable motions of a rigid body

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Abstract. An equivalence is established between the problem of motion of a rigid body with spherical dynamical symmetry about a fixed point under the action of potential and gyroscopic forces and the problem of motion of a particle on a smooth fixed sphere. A new integrable case of the first problem is constructed from a case of the second problem, recently found by Gaffet. Three integrable cases of the second problem are obtained from the known integrable cases in various problems of rigid-body dynamics.

1. Introduction

Integrable problems are rare exceptions in the totality of problems in mechanics. For them one can make many important assertions about the global behaviour of motion and in some cases the general explicit time solution of the equations of motion can be obtained. Integrable systems are also of great importance in the study of nonintegrable systems near to them. The quest for integrable mechanical systems remains one of the principal fields of investigation. As there is no general method for deciding about the integrability of a given dynamical problem, it is essential to tabulate all integrable problems found by diverse methods. This is even more essential in rigid-body dynamics, where the usual method of trying some ansatz of an integral of motion leads to tremendous difficulties. A natural way of constructing new integrable cases is the generalization of known ones by introducing additional physical parameters. Remarkable examples are the cases presented in [1, 2]. Another way is geometrically transforming the problem of motion of a rigid body under a certain set of forces to another problem of motion of a rigid body under a different set of forces, so that integrable cases of one problem transform to integrable cases of the other. Various examples of this type are given in [3, 4].

Interesting as well are integrable problems of motion of a particle on a fixed smooth surface under various types of force. Integrable motions on the sphere, separable in the sphero-conical coordinates, are classical examples (e.g. [5, 6]). Separable systems on the sphere were used to generate conditional integrable problems of motion of rigid bodies (e.g. [7–10]). The motion on a sphere is also related to other diverse physical systems. For example, it is met in the study of the B-phase of the superfluid ^{3}He, in the construction of certain wave solutions of the Landau–Lifshitz nonlinear equation (e.g. [11]) and in the treatment of Dyson's fluid dynamical model of spinning gas clouds maintaining ellipsoidal shape [12].

In this paper we use the isomorphism between the problem of motion of a rigid body with complete dynamical symmetry acted upon by potential and gyroscopic forces and that of motion of a particle on a smooth fixed sphere to point out new integrable cases of each problem using known cases of the other.

Let the body have spherical dynamical symmetry with respect to the fixed point. Without loss of generality we can take $A = B = C = 1$. Let that body be in motion under the action of potential and gyroscopic forces so that it is characterized by the Lagrangian

$$L = \tfrac{1}{2}\boldsymbol{\omega}^2 + \boldsymbol{l}\cdot\boldsymbol{\omega} - V(\boldsymbol{\gamma}) \tag{1}$$

where $\boldsymbol{\omega} = (p, q, r) = (\dot\psi \sin\theta \sin\varphi + \dot\theta \cos\varphi, \dot\psi \sin\theta \cos\varphi - \dot\theta \sin\varphi, \dot\psi \cos\theta)$ and $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \gamma_3) = (\sin\theta \sin\varphi, \sin\theta \cos\varphi, \cos\theta)$ denote the angular velocity and the unit vector fixed in space, both referred, as usual, to the body system, the functions $\boldsymbol{l}$, V are vector and scalar functions of $\boldsymbol{\gamma}$ and ψ, θ, φ are Euler's angles of precession, nutation and proper rotation, respectively. The Euler–Poisson equations are (see e.g. [13])

$$\dot{\boldsymbol{\omega}} + \boldsymbol{\omega}\times\boldsymbol{\mu} = \boldsymbol{\gamma}\times\frac{\partial V}{\partial\boldsymbol{\gamma}} \qquad \dot{\boldsymbol{\gamma}} + \boldsymbol{\omega}\times\boldsymbol{\gamma} = \mathbf{0} \tag{2}$$

where

$$\boldsymbol{\mu} = \frac{\partial}{\partial\boldsymbol{\gamma}}(\boldsymbol{\gamma}\cdot\boldsymbol{l}) - \left(\frac{\partial}{\partial\boldsymbol{\gamma}}\cdot\boldsymbol{l}\right)\boldsymbol{\gamma}. \tag{3}$$

This system admits the cyclic integral

$$(\boldsymbol{\omega} + \boldsymbol{l})\cdot\boldsymbol{\gamma} = f \tag{4}$$

corresponding to the angle of precession ψ and a geometric integral

$$|\boldsymbol{\gamma}|^2 = 1. \tag{5}$$

From the integral (4) we can obtain

$$\dot\psi = f - \boldsymbol{l}\cdot\boldsymbol{\gamma} - \frac{\gamma_3(\gamma_2\dot\gamma_1 - \gamma_1\dot\gamma_2)}{\gamma_1^2 + \gamma_2^2}. \tag{6}$$

We first multiply Poisson's equation (the second equation in (2)) vectorially by $\boldsymbol{\gamma}$ to obtain

$$\boldsymbol{\gamma}\times\dot{\boldsymbol{\gamma}} + |\boldsymbol{\gamma}|^2\boldsymbol{\omega} - (\boldsymbol{\omega}\cdot\boldsymbol{\gamma})\boldsymbol{\gamma} = \mathbf{0} \tag{7}$$

and then use (4) and (5) to express the angular velocity in the form

$$\boldsymbol{\omega} = \dot{\boldsymbol{\gamma}}\times\boldsymbol{\gamma} + (f - \boldsymbol{l}\cdot\boldsymbol{\gamma})\boldsymbol{\gamma}. \tag{8}$$

As in [10], the cyclic integral can be used to ignore the cyclic variable using the components of the vector $\boldsymbol{\gamma}$ as configurational variables subject to the constraint 5. We obtain the Routhian

$$\begin{aligned} R = L - f\dot\psi = {}& \frac{1}{2}(\dot\gamma_1^2 + \dot\gamma_2^2 + \dot\gamma_3^2) + \frac{f\gamma_3}{\gamma_1^2 + \gamma_2^2}(\gamma_2\dot\gamma_1 - \gamma_1\dot\gamma_2) + (\boldsymbol{\gamma}\times\boldsymbol{l})\cdot\dot{\boldsymbol{\gamma}} \\ & - \left[V + \frac{(f - \boldsymbol{l}\cdot\boldsymbol{\gamma})^2}{2}\right]. \end{aligned} \tag{9}$$

Consider now the second problem: a particle of unit mass moves at the current point $\boldsymbol{r} = (x, y, z)$ on the smooth fixed sphere

$$|\boldsymbol{r}|^2 = 1 \tag{10}$$

under the action of potential and gyroscopic forces with scalar and vector potentials V_p and $\boldsymbol{A}$ respectively:

$$L = \tfrac{1}{2}\dot{\boldsymbol{r}}^2 + \boldsymbol{A}\cdot\dot{\boldsymbol{r}} - V_p. \tag{11}$$

This system has two degrees of freedom and admits the integral

$$E = \tfrac{1}{2}\dot{\boldsymbol{r}}^2 + V_p. \tag{12}$$

It will be completely integrable in the sense of Liouville whenever we can find one more independent integral.

The two Lagrangian systems (5), (9) and (10), (11) can be made completely identical if we choose

$$\begin{aligned} V_p &= V + \frac{(f - \boldsymbol{l}' \cdot \boldsymbol{r})^2}{2} \\ \boldsymbol{A} &= \boldsymbol{r} \times \left[\boldsymbol{l}' + \frac{fz}{x^2 + y^2} (0, 0, 1) \right] \end{aligned} \tag{13}$$

in which f enters as a parameter and $\boldsymbol{l}'$ is obtained from $\boldsymbol{l}$ through replacing $\boldsymbol{\gamma}$ by $\boldsymbol{r}$. This establishes the equivalence of the problem of motion of the body under consideration on a fixed level f of the cyclic integral and the problem of motion of a particle on a sphere. In particular, an integrable case of one of them immediately leads to an integrable case of the other. We now use this situation to update the status of the two problems as concerns the integrable cases in each of them.

Although in applications the vector $\boldsymbol{A}$ in (11) can arise due to various physical effects, it is convenient to think of it as the vector potential of a magnetic field†, which can be expressed as $\boldsymbol{H} = \mathrm{curl}\boldsymbol{A}$. It is evident that the vector potential is not uniquely determined for a given problem. A term of the form $\nabla\Phi(\boldsymbol{r})$ can always be added to it without changing the Lorentz forces in the equations of motion. On the other hand, only the radial component $H_r = \boldsymbol{H} \cdot \boldsymbol{r}$ of the magnetic field will contribute to the equations of motion on the sphere. Thus, the problem is completely determined by the two scalar functions V_p and H_r if given on the sphere. We now write the equations of motion in such a way that only those two functions appear in them.

The Lagrangian equations of motion of the system (11) under the constraint (10) have the form

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{\boldsymbol{r}}} - \frac{\partial L}{\partial \boldsymbol{r}} = \lambda \boldsymbol{r} \tag{14}$$

where λ is a scalar multiplier. This gives

$$\ddot{\boldsymbol{r}} = \dot{\boldsymbol{r}} \times \boldsymbol{H} - \frac{\partial V}{\partial \boldsymbol{r}} + \lambda \boldsymbol{r}. \tag{15}$$

Multiplying the last equation vectorially by $\boldsymbol{r}$ and noting that $\boldsymbol{r} \cdot \dot{\boldsymbol{r}} = 0$, we eliminate λ and obtain the final equations

$$\boldsymbol{r} \times \ddot{\boldsymbol{r}} = H_r \dot{\boldsymbol{r}} - \boldsymbol{r} \times \frac{\partial V}{\partial \boldsymbol{r}}. \tag{16}$$

We note now that the term of $\boldsymbol{A}$ in (13) which depends on f contributes a constant value f to H_r. An interpretation of the gyroscopic forces is possible as due to the Lorentz effect of a fixed magnetic pole of strength f at the centre of the sphere on a unit charge carried by the particle.

2. A new integrable problem in rigid body dynamics

In [12, 14], Gaffet has established the integrability of problem (10), (11) for the choice

$$V_p = \frac{K}{(xyz)^{\frac{2}{3}}} \qquad \boldsymbol{A} = 0. \tag{17}$$

† Here MKS units are used. In Gaussian units the linear term in (11) should be divided by the velocity of light c. We also assume that the velocity and acceleration are sufficiently small to neglect both relativistic effects and classical radiation damping.

A possible choice in the corresponding rigid-body problem is

$$V = \frac{K}{(\gamma_1\gamma_2\gamma_3)^{\frac{2}{3}}} \qquad \boldsymbol{l} = \mathbf{0} \qquad f = 0. \tag{18}$$

In fact, it is easy to verify that the equations of motion (2) on the cyclic integral level

$$A(p\gamma_1 + q\gamma_2 + r\gamma_3) = 0 \tag{19}$$

admit the cubic integral

$$Apqr - 2K(\gamma_1\gamma_2\gamma_3)^{\frac{1}{3}}\left(\frac{p}{\gamma_1} + \frac{q}{\gamma_2} + \frac{r}{\gamma_3}\right). \tag{20}$$

This adds a new case to the list of eight known conditional integrable cases of the rigid-body dynamics under axisymmetric forces (see [2], table 2). This new case has the unique feature that its potential becomes singular when any one of the three principal axes of the body reaches the plane orthogonal to the vector $\boldsymbol{\gamma}$. Other cases are known in which the potential becomes singular when the following hold.

(1) An axis of the body coincides with the vector $\boldsymbol{\gamma}$. Examples are case 5 of table 1 and case 7 of table 2 in [2].
(2) One axis of the body reaches the plane orthogonal to the vector $\boldsymbol{\gamma}$. Examples are cases 1, 2 and 4 of table 2 [2].
(3) Both types of singularity are present (case 3 of table 2 [2]).

It is to be noted that (18) is just the simplest choice of $V, \boldsymbol{l}, f$. In fact, to determine all possible choices one has to solve the relations (13) in $V, \boldsymbol{l}$. The solution will involve f as a parameter and the resulting vector $\boldsymbol{l}$ will always contain a term $\nu(\boldsymbol{\gamma})\boldsymbol{\gamma}$, depending on an arbitrary scalar function $\nu(\boldsymbol{\gamma})$. The choice (18) characterizes the basic case of the integrable class of problems in the sense of [2].

3. Integrable cases of the particle on a sphere

In addition to the above new case, there are four known basic integrable cases of a rigid body with spherical dynamical symmetry. They are presented in [2]: cases 2 and 3 in table 1 and cases 6 and 7 in table 2. Case 6 of table 2 corresponds to potentials on the sphere separable in the sphero-conical coordinates. The other three cases correspond to nonseparable integrable cases of (10), (11), which we list as follows.

(1) *The case corresponding to Clebsch's case.* This is characterized by the pair of functions

$$\begin{aligned} V_p &= ax^2 + by^2 + cz^2 \\ H_r &= f. \end{aligned} \tag{21}$$

The second integral of motion for this case can be obtained from Clebch's integral, substituting $\boldsymbol{\omega} \to f\boldsymbol{r} - \boldsymbol{r} \times \dot{\boldsymbol{r}}$ (compare with (8)).

$$\begin{aligned} I = {} & a(y\dot{z} - z\dot{y} - fx)^2 + b(z\dot{x} - x\dot{z} - fy)^2 + c(x\dot{y} - y\dot{x} - fz)^2 \\ & -(bcx^2 + cay^2 + abz^2). \end{aligned} \tag{22}$$

This case is a nonseparable generalization of the well known separable Neumann integrable problem [5] by the presence of the gyroscopic forces and reduces to it when $f = 0$.

(2) *The case corresponding to the Rubanovsky–Lyapunov case*

$$V_p = s_1 x + s_2 y + s_3 z - \frac{abc}{2}\left(\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c}\right)$$
$$+\tfrac{1}{8}[2f + (b+c)x^2 + (c+a)y^2 + (a+b)z^2]^2$$
$$H_r = f + \tfrac{1}{2}[a+b+c-3(ax^2+by^2+cz^2)]. \tag{23}$$

The second integral of motion is

$$I = (b+c)(y\dot{z} - z\dot{y} - Nx)^2 + (c+a)(z\dot{x} - x\dot{z} - Ny)^2$$
$$+(a+b)(x\dot{y} - y\dot{x} - Nz)^2 + s_1[(N+a)x + z\dot{y} - y\dot{z}]$$
$$+s_2[(N+b)y + x\dot{z} - z\dot{x}] + s_3[(N+c)z + y\dot{x} - x\dot{y}]$$
$$-(bcx^2 + cay^2 + abz^2) \tag{24}$$

where $N = f + \frac{1}{2}[(b+c)x^2 + (c+a)y^2 + (a+b)z^2]$.

(3) *The case corresponding to case 7 of table 2*

$$V_p = v + \frac{K}{\sqrt{v}}$$
$$H_r = \tfrac{1}{2}[a+b+c-3(ax^2+by^2+cz^2)] \tag{25}$$

where $v = \frac{1}{8}[(b+c)x^2 + (c+a)y^2 + (a+b)z^2]^2 - \frac{1}{2}abc(\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c})$ and K is a constant. This admits the complementary integral

$$I = \tfrac{1}{2}[(b+c)(y\dot{z} - z\dot{y} - Nx)^2 + (c+a)(z\dot{x} - x\dot{z} - Ny)^2 + (a+b)(x\dot{y} - y\dot{x} - Nz)^2]$$
$$-(bcx^2 + cay^2 + abz^2) + K\frac{a+b+c+ax^2+by^2+cz^2}{2\sqrt{v}}. \tag{26}$$

References

[1] Yehia H M 1997 New generalizations of the integrable problems in rigid body dynamics *J. Phys. A: Math. Gen.* **30** 7269–75

[2] Yehia H M 1999 New generalizations of all the known integrable problems in rigid body dynamics *J. Phys. A: Math. Gen.* **32** 7565–80

[3] Yehia H M 1998 New integrable problems of motion of a rigid body acted upon by nonsymmetric electromagnetic forces *J. Phys. A: Math. Gen.* **31** 5819–25

[4] Yehia H M Geometric transformations and new integrable problems in rigid body dynamics 2000 *J. Phys. A: Math. Gen.* **33** 4393–9

[5] Neumann C 1859 De problemate quodam mechanico, quod ad primam integralium ultraellipticorum classen revocatur *Reine Angew. Math.* **56** 46–63

[6] Olsson O I 1909 Tillämpning af de hyperelliptiska funktionerna inom den materiella punktens dynamik *Ark. Mat. Astr. Fys.* **6** 1–27

[7] Kozlov V V 1982 Hamiltonian equations of the problem of motion of a rigid body with a fixed point in redundant coordinates *Teor. Primen. Meh.* **8** 59–65

[8] Bogoyavlensky O I 1986 Integrable cases of the dynamics of a rigid body and integrable systems on the spheres S^n *Math. USSR Izv.* **27** 203–18

[9] Yehia H M 1986 On the integrability of certain problems in particle and rigid body dynamics *J. Mecan. Theor. Appl.* **5** 55–71

[10] Yehia H M 1988 Equivalent problems in rigid body dynamics—I *Celest. Mech.* **41** 275–88

[11] Dubrovin B A, Krichever I M and Novikov S P 1986 Integrable systems I *Dynamical Systems IV, Symplectic Geometry and its Applications* (Berlin: Springer)

[12] Gaffet B 1998 A completely integrable Hamiltonian motion on the surface of a sphere *J. Phys. A: Math. Gen.* **31** 1581–96

[13] Yehia H M 1986 On the motion of a rigid body acted upon by potential and gyroscopic forces. I. The equations of motion and their transformations *J. Mecan. Theor. Appl.* **5** 747–54.

[14] Gaffet B 1998 An integrable Hamiltonian motion on a sphere II. The separation of variables *J. Phys. A: Math. Gen.* **31** 8341–54